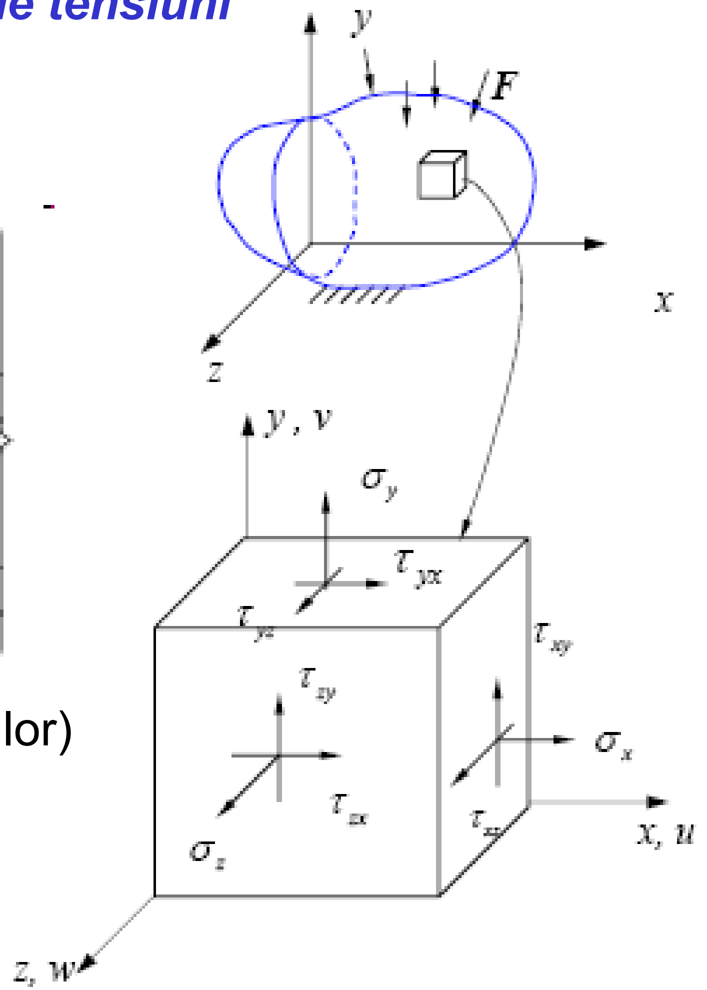
4. Elemente de volum pentru probleme 3-D

4.1 Elemente de teoria elasticității 3-D

Starea spațială de tensiuni

$$\boldsymbol{\sigma}=\{\sigma\}=\begin{Bmatrix}\sigma_x\\ \sigma_y\\ \sigma_z\\ \tau_{xy}\\ \tau_{yz}\\ \tau_{zx}\end{Bmatrix}$$

(vectorul tensiunilor)



Starea spațială de deformație

$$\varepsilon = \{\varepsilon\} = \begin{Bmatrix} \varepsilon_x \\ \varepsilon_y \\ \varepsilon_z \\ \gamma_{xy} \\ \gamma_{yz} \\ \gamma_{zx} \end{Bmatrix}$$

(vectorul deformațiilor)

Relația dintre tensiuni şi deformații (relații constitutive)

$$\begin{Bmatrix} \sigma_x \\ \sigma_y \\ \sigma_z \\ \tau_{xy} \\ \tau_{yz} \\ \tau_{zx} \end{Bmatrix} = \frac{E}{(1+\nu)(1-2\nu)} \begin{bmatrix} 1-\nu & \nu & \nu & 0 & 0 & 0 \\ \nu & 1-\nu & \nu & 0 & 0 & 0 \\ \nu & \nu & 1-\nu & 0 & 0 & 0 \\ 0 & 0 & 0 & \frac{1-2\nu}{2} & 0 & 0 \\ 0 & 0 & 0 & 0 & \frac{1-2\nu}{2} & 0 \\ 0 & 0 & 0 & 0 & 0 & \frac{1-2\nu}{2} \end{bmatrix} \begin{Bmatrix} \varepsilon_x \\ \varepsilon_y \\ \varepsilon_z \\ \gamma_{xy} \\ \gamma_{yz} \\ \gamma_{zx} \end{Bmatrix}$$

sau condensat: $$\boldsymbol{\sigma} = \mathbf{E}\boldsymbol{\varepsilon}$$

Depasări ale unui punct

$$\mathbf{u} = \begin{Bmatrix} u(x,y,z) \\ v(x,y,z) \\ w(x,y,z) \end{Bmatrix} = \begin{Bmatrix} u_1 \\ u_2 \\ u_3 \end{Bmatrix}$$

Relața dintre deformații specifice şi deplasări

$$\varepsilon_x = \frac{\partial u}{\partial x}, \quad \varepsilon_y = \frac{\partial v}{\partial y}, \quad \varepsilon_z = \frac{\partial w}{\partial z},$$

$$\gamma_{xy} = \frac{\partial v}{\partial x} + \frac{\partial u}{\partial y}, \quad \gamma_{yz} = \frac{\partial w}{\partial y} + \frac{\partial v}{\partial z}, \quad \gamma_{xz} = \frac{\partial u}{\partial z} + \frac{\partial w}{\partial x}$$

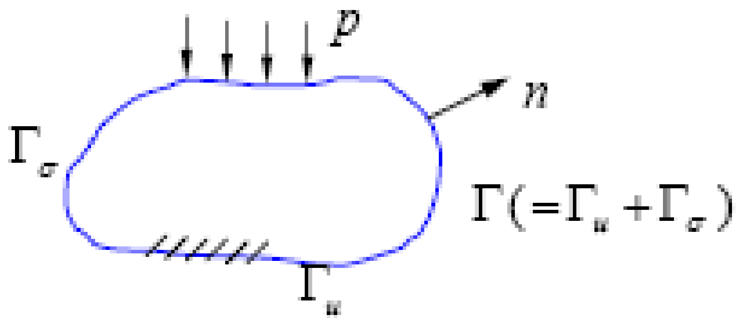
Ecuații de echilibru

$$\frac{\partial \sigma_x}{\partial x} + \frac{\partial \tau_{xy}}{\partial y} + \frac{\partial \tau_{xz}}{\partial z} + f_x = 0 \ ,$$

$$\frac{\partial \tau_{yx}}{\partial x} + \frac{\partial \sigma_y}{\partial y} + \frac{\partial \tau_{yz}}{\partial z} + f_y = 0 \ ,$$

$$\frac{\partial \tau_{zx}}{\partial x} + \frac{\partial \tau_{zy}}{\partial y} + \frac{\partial \sigma_z}{\partial z} + f_z = 0 \ ,$$

Condiții de margine



$u_i=\bar{u}_i,$ pe Γ_u (deplasări cu valori cunoscute)

$t_i=\bar{t}_i,$ pe Γ_σ (forțe cunoscute)

Analiza stărilor de eforturi

Probleme 3-D implcă ***15 necunoscute*** (6 tensiuni, 6 deformații şi 3 deplasări).

Ecuații disponibile = 15 (6 relații constitutive, 6 relații între deformații şi deplasări şi 3 ecuații de echilibru).

Soluții analitice dificil de obținut....

4.2 Elemente finite pentru probleme 3-D

Câmpul deplasărilor (depasările unui punct din interiorul elementului)

$$u = \sum_{i=1}^{N} N_i u_i$$

$$v = \sum_{i=1}^{N} N_i v_i$$

$$w = \sum_{i=1}^{N} N_i w_i$$

deplasări ale nodurilor elementului

sau în formă matriceală

$$\begin{Bmatrix} u \\ v \\ w \end{Bmatrix}_{(3\times1)} = \begin{bmatrix} N_1 & 0 & 0 & N_2 & 0 & 0 & \cdots \\ 0 & N_1 & 0 & 0 & N_2 & 0 & \cdots \\ 0 & 0 & N_1 & 0 & 0 & N_2 & \cdots \end{bmatrix}_{(3\times3N)} \begin{Bmatrix} u_1 \\ v_1 \\ w_1 \\ u_2 \\ v_2 \\ w_2 \\ \vdots \end{Bmatrix}_{(3N\times1)}$$

condensat:

$$\mathbf{u} = \mathbf{N}\,\mathbf{d}$$

Vectorul deformațiilor se obține pe baza relațiilor dintre dintre deformații şi deplasări si a expresiei câmpului deplasărilor:

$$\boldsymbol{\varepsilon} = \mathbf{B}\,\mathbf{d}$$

$$(6\times1) \quad (6\times3N)\times(3N\times1)$$

Matricea de rigiditate

$$\mathbf{k} = \int_v \mathbf{B}^T \mathbf{E}\,\mathbf{B}\,dv$$

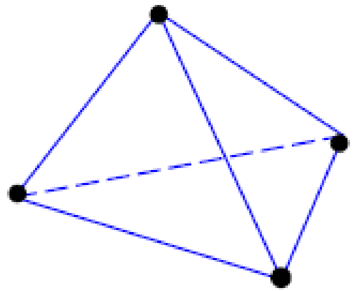
$$(3\times\mathrm{N}) \quad (3N\times6)\times(6\times6)\times(6\times3N)$$

În practica curentă evaluarea integralei se realizaează prin procedee numerice.

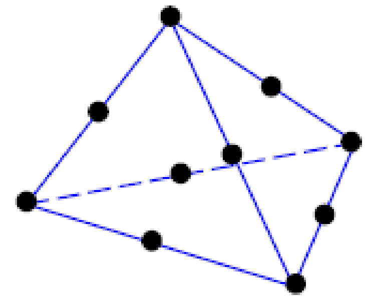
Matricea de rigiditate conține deplasările de corp rigid ale elementului (3 translații şi 3 rotații). Ele vor fi eliminate la nivelul modelului cu EF.

4.3 Tipuri de elemente finite 3-D

Tetraedre

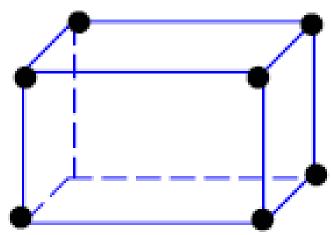


liniar – 3 noduri

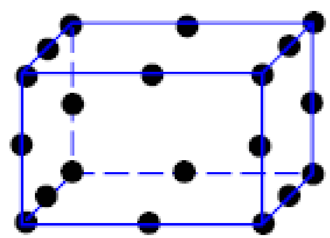


pătratic – 10 noduri

Hexaedre

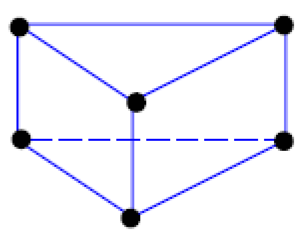


liniar – 8 noduri

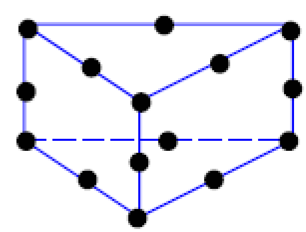


pătratic – 20 noduri

Pentaedre

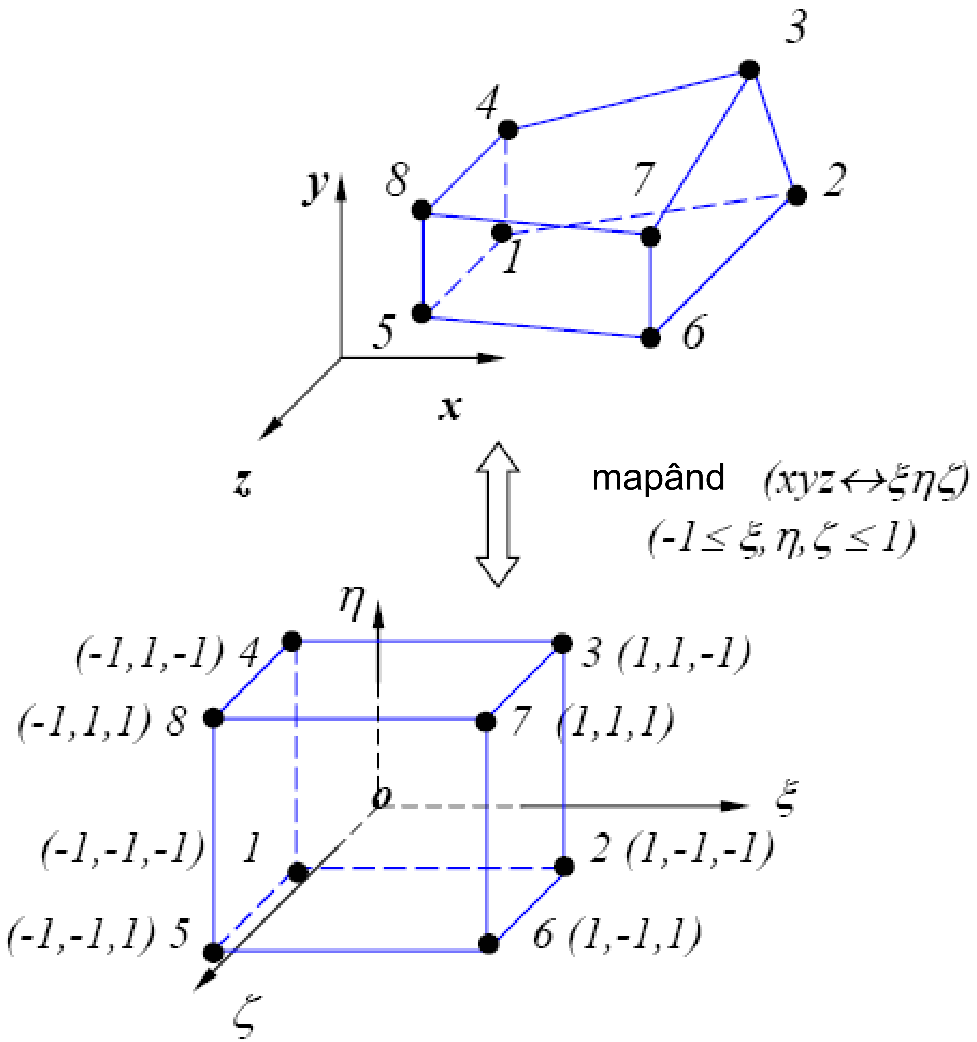


liniar – 6 noduri



pătratic – 15 noduri

Formularea elementului hexaedru liniar



Câmpul depasărilor în interiorul elementului:

$$u = \sum_{i=1}^{8} N_i u_i \quad , \qquad v = \sum_{i=1}^{8} N_i v_i \quad , \quad w = \sum_{1i=1}^{8} N_i w_i$$

Funcții de interpolare (de formă):

$$N_1(\xi,\eta,\zeta)=\frac{1}{8}(1-\xi)(1-\eta)(1-\zeta) \quad ,$$

$$N_2(\xi,\eta,\zeta)=\frac{1}{8}(1+\xi)(1-\eta)(1-\zeta) \quad ,$$

$$N_3(\xi,\eta,\zeta)=\frac{1}{8}(1+\xi)(1+\eta)(1-\zeta) \quad ,$$

$$\vdots \qquad\qquad\qquad \vdots$$

$$N_8(\xi,\eta,\zeta)=\frac{1}{8}(1-\xi)(1+\eta)(1+\zeta) \quad .$$

Propietăți ale finctiilor de formă:

$$N_i(\xi_j,\eta_j,\zeta_j)=\delta_{ij} \quad , \quad i,j=1,2,\cdots,8.$$

$$\sum_{i=1}^{8} N_i(\xi,\eta,\zeta)=1.$$

Transformarea coordonatelor (mapare):

$$x=\sum_{i=1}^{8} N_i\, x_i \quad , \quad y=\sum_{i=1}^{8} N_i\, y_i \quad , \quad z=\sum_{i=1}^{8} N_i\, z_i \quad .$$

Daca se utilizează **aceleaşi funcţii de interpolare** pentru transformarea coordonatelor ca cele folosite la interpolarea câmpului deplasărilor – rezultă **element izoparametric.**

Deformaţiil specifice

$$\boldsymbol{\varepsilon}=\begin{Bmatrix}\varepsilon_x\\ \varepsilon_y\\ \varepsilon_z\\ \gamma_{xy}\\ \gamma_{yz}\\ \gamma_{zx}\end{Bmatrix}=\begin{Bmatrix}\dfrac{\partial u}{\partial x}\\ \dfrac{\partial v}{\partial y}\\ \dfrac{\partial w}{\partial z}\\ \dfrac{\partial x}{\partial x}+\dfrac{\partial u}{\partial y}\\ \dfrac{\partial w}{\partial y}+\dfrac{\partial v}{\partial z}\\ \dfrac{\partial u}{\partial z}+\dfrac{\partial w}{\partial x}\end{Bmatrix}=\mathbf{B\,d}$$

$$\underset{(6\times 1)}{\boldsymbol{\varepsilon}}=\underset{(6\times 24)\times(24\times 1)}{\mathbf{B\,d}}$$

Matricea Jacobi

$$\begin{Bmatrix} \dfrac{\partial u}{\partial \xi} \\ \dfrac{\partial u}{\partial \eta} \\ \dfrac{\partial u}{\partial \zeta} \end{Bmatrix} = \underbrace{\begin{bmatrix} \dfrac{\partial x}{\partial \xi} & \dfrac{\partial y}{\partial \xi} & \dfrac{\partial z}{\partial \xi} \\ \dfrac{\partial x}{\partial \eta} & \dfrac{\partial y}{\partial \eta} & \dfrac{\partial z}{\partial \eta} \\ \dfrac{\partial x}{\partial \zeta} & \dfrac{\partial y}{\partial \zeta} & \dfrac{\partial z}{\partial \zeta} \end{bmatrix}}_{\equiv \mathbf{J}} \begin{Bmatrix} \dfrac{\partial u}{\partial x} \\ \dfrac{\partial u}{\partial y} \\ \dfrac{\partial u}{\partial z} \end{Bmatrix}$$

matricea Jacobi

Prin inversarea relației rezultă:

$$\begin{Bmatrix} \dfrac{\partial u}{\partial x} \\ \dfrac{\partial u}{\partial y} \\ \dfrac{\partial u}{\partial z} \end{Bmatrix} = \mathbf{J}^{-1} \begin{Bmatrix} \dfrac{\partial u}{\partial \xi} \\ \dfrac{\partial u}{\partial \eta} \\ \dfrac{\partial u}{\partial \zeta} \end{Bmatrix}, \quad \left(\frac{\partial u}{\partial \xi} = \sum_{i=1}^{8} \frac{\partial N_i}{\partial \xi} u_i \quad , etc. \right)$$

Similar se obțin:

$$\begin{Bmatrix} \dfrac{\partial v}{\partial x} \\ \dfrac{\partial v}{\partial y} \\ \dfrac{\partial v}{\partial z} \end{Bmatrix} = \mathbf{J}^{-1} \begin{Bmatrix} \dfrac{\partial v}{\partial \xi} \\ \dfrac{\partial v}{\partial \eta} \\ \dfrac{\partial v}{\partial \zeta} \end{Bmatrix}$$ şi pentru: *w*

Energia de deformare

$$U = \frac{1}{2}\int_V \boldsymbol{\sigma}^T \boldsymbol{\varepsilon}\, dV = \frac{1}{2}\int_V (\mathbf{E}\boldsymbol{\varepsilon})^T \boldsymbol{\varepsilon}\, dV = \frac{1}{2}\int_V \boldsymbol{\varepsilon}^T \mathbf{E}\, \boldsymbol{\varepsilon}\, dV$$

$$= \frac{1}{2}\mathbf{d}^T \left[\int_V \mathbf{B}^T \mathbf{E}\, \mathbf{B}\, dV \right] \mathbf{d}$$

Matricea de rigiditate

$$\mathbf{k} = \int_V \mathbf{B}^T \mathbf{E}\, \mathbf{B}\; dV$$

$$(24\times 24) \quad (24\times 6)\times(6\times 6)\times(6\times 24)$$

Pentru exactitatea şi simplitate în calcule, acestea se realizează în coordontate locale $\xi\eta\zeta$

$$dV = (\det \mathbf{J})\, d\xi\, d\eta\, d\zeta$$

$$\mathbf{k} = \int_{-1}^{1}\int_{-1}^{1}\int_{-1}^{1} \mathbf{B}^T \mathbf{E}\, \mathbf{B}\, (\det \mathbf{J})\, d\xi\, d\eta\, d\zeta$$

Integrare numerică

Evaluarea tensiunilor

$$\boldsymbol{\sigma} = \mathbf{E}\,\boldsymbol{\varepsilon} = \mathbf{E}\,\mathbf{B}\,\mathbf{d}$$

Tensiuni principale

$$\sigma_1, \sigma_2, \sigma_3 .$$

Tensiuni von Mises

$$\sigma_e = \sigma_{VM} = \frac{1}{\sqrt{2}}\sqrt{(\sigma_1 - \sigma_2)^2 + (\sigma_2 - \sigma_3)^2 + (\sigma_3 - \sigma_1)^2}\ .$$

Tensiunile se evaluează la nodurile şi punctele de integrare ale elementului.

Tensiunile pot fi mediate la nodurile comune mai multor elemente.